VIRTUAL MANIPULATIVES IN MATHEMATICS EDUCATION: A THEORETICAL FRAMEWORK*

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Abstract: *Meaningful educational activities and cognitive tools might improve students' active involvements in the teaching-learning process and encourage their reflections on the concepts and relations to be investigated. It is claimed that usage of manipulatives not only increase students' conceptual understanding and problem solving skills but also promotes their positive attitudes towards mathematics since they supposedly provide "concrete experiences" that focus attention and increase motivation. A concrete experience in mathematics context is defined not by its physical or real-world characteristics but rather by how meaningful connections it could make with other mathematical ideas and situations. For instance, a student might create the meaning of the concept "four" by building a representation of the number and connecting it with either real or pictured blocks. Computer manipulatives, also called virtual manipulatives, may provide interactive environments where students could pose and solve their own problems to form connections between mathematical concepts and operations, and get immediate feedback about their actions. Hence, it is necessary to design specific math manipulatives focussing at different mathematical concepts. Virtual manipulatives might also provide further advantages over physical manipulatives by eliminating some of the constraints they impose on the task. In this paper, virtual manipulatives in mathematics education will be introduced, their main characteristics will be explained and the implications of the usage of virtual manipulatives in mathematics classrooms will be thoroughly discussed.*
Keywrords*: virtual manipulative, mathematical abstraction, modeling*

INTRODUCTION

Mathematicians have used several tools, such as sliding rules, compass, calculators and recently computers, to simplify doing mathematics throughout history. However, employing tools in an education requires paying special attention to certain pedagogical concerns. Hence, the provision of tools is not just sufficient without clarifying adequately its place and the usage policy in the teaching-learning process. For instance, the computer, from the very beginning of its invention, has taken its place in education. Computers made life easier for mathematics educators and people doing mathematics with the help of several software packages capable of word-processing and making difficult mathematical calculations and drawings. Employment of computers in math classrooms became synonymous with learning how to use those software packages to simplify mathematical calculations such as Mathematica, Derive and MathCad. After computers became ubiquitous and affordable, attention soon shifted from "learning to use computers to do math" to "using computers as an aid in a math lesson". Earlier applications considered the computer as another medium to display and test the content material in the form of programmed instruction (Skinner, 1954) and intelligent tutoring systems (Koedinger et al. 1997). These systems mainly adopted drill and practice approach, advocated strict control over instructional method employed and the content material presented and generally hold the intrinsic view that the computer could become someday a good replacement for books and teachers to some extent. However, skeptical educators especially holding constructivist views opposed this approach and redefined the computer's role as a tool enabling free explorations of the concepts and relations in open ended tasks void of any instructional method and content. Several software packages, called microworlds, were implemented to enable explorations in math. Logo and dynamic geometry software applications such as Cabri and Sketchpad, are the most widely used and prominent of this kind. Incorporation of these packages into mathematics lesson required specific teaching activities and a large collection of activities accumulated over the years. Hence, computers' place and functionality in an educational context nowadays could best be described with a "cognitive tool" metaphor that supports cognitive apprenticeship by scaffolding the important processes of articulation and reflection that are the foundations of knowledge construction (Collins et al., 1989). Salomon et al. (1991) describes learning with computers as the mindful engagement of learners in the tasks afforded by the computer, i.e., an intellectual partnership with the computer. Norman (1993) also argues that computers support reflective thinking which is defined as the careful, deliberate kind of thinking that helps us not only make sense out of what we have experienced and what we know but also to compose new knowledge by adding new representations, modifying old ones, and comparing the two. Educators holding socio-cultural constructivist views may still be cautious about these applications since they are not designed in a way to support collaborative and cooperative learning strategies. However, computers role as a thought-provoking tool seems to be firm among educators whatever view they may hold. In fact, mathematics itself could be considered as a tool for problem solving and organizing one's thinking through mathematical modeling.

MATHEMATICAL MODELING

Mathematics is often seen as an isolated experience area performed just in schools alienated from real life. In fact, mathematics is a systematic way of thinking that produce solutions to problems by modeling real-world situations. *Modeling* could be defined as translating a problem at hand into mathematical notations, i.e., describing it in a mathematical language, by seeing mathematics as a tool for problem solving. In fact, all mathematical concepts have roots in the real world. A situation could be translated into the mathematical symbols in order to enable mathematical calculations. For example, the problem of bringing together two sets of sheeps having three and four sheeps respectively could be translated into mathematical symbols as 3+4 and the result is found as 7 seven sheeps. The process of mathematical modeling consists of three main stages; formulation of a real world situation as a mathematical problem (creation of a mathematical model), the mathematical solution of the problem and finally translating back the solution into the original context in order to interpret the results produced by the model to help solve the real problem (Berry & Houston, 1995). If the model acts in a way that truly parallels the original, then it becomes feasible to manipulate and employ the model to make predictions and conclusions about its counterpart in the real world (Post, 1981). Modeling is a way of simplifying the real world problems by making abstractions. Abstraction in turn is to reach a much more simple depiction of a system by deciding on the most significant elements and the salient features of the system and omitting other elements and features of minor importance. The aim of mathematical modeling, then, becomes to understand, to explain, to describe and to predict the different aspects of the real world. By the help of the mathematical models, we could enrich our understanding of the concepts and relations and learn how to control some aspects of the systems by predicting how their objects will behave under certain circumstances. For instance, the ancient Egyptians used geometry to model land problems and improve irrigation and astronomers used mathematical model in order to be able to accurately predict the motion of the planets.

Modeling might be used both as a teaching and assessment tool since mathematical models might be viewed as external indicators of student cognitive structures that are built and amplified through the tutor's interventions and the most important goal of teaching mathematics is to instill a value of the possibilities of using mathematical methods to handle incoming problems from all different parts of life (Duncan et al., 1996). The initial steps of mathematical modeling require identification of adequate and appropriate representations of the objects in the problem situation. Representations are interpretations of the reality. Mathematical concepts and relationships could be exemplified through these representations. Mathematical representations could help students recognize connections among related concepts and improve their communication skills in mathematics. Multiple representations, such as diagrams, graphical displays, and symbolic expressions, are also important to convey the various aspects of the same mathematical concept. However, representations, no matter how concrete they are, often does not serve the purpose of clarifying concepts if they are perceived as an end-product rather than as a tool to interpret the reality.

There are two different approaches in using models in learning environments; “Learning to model“ and “learning with models”. Learning to model approach advocates teaching how to model the reality. Learners are expected to construct their own models and models are used as a communication medium to express learner’s knowledge. Although microworlds such as Logo and Cabri could be regarded as adopting this approach to some extent, using computer as a tool to create novel models is not easy. For instance, Cabri geometry enables learners to make their own constructions and models. However, there is no way to check or to verify the consequences of the model. Likewise, one could solve certain mathematical problems with the help of Mathematica or other computer algebra systems but there is no mechanism to enter a model to be evaluated by these systems. This approach requires learners to have a significant understanding of the underlying objects of the model and could be regarded as the end product of an educational process rather than being used certain while concepts are trying to be conveyed.

Learning with models approach, on the contrary, encourage learners to solve problems by the help of ready-made models.

In fact, Simon (1981) argues that solving a problem simply means representing it so as the solution is transparent. Learners are given ready models specially created for certain problems or situations and are required to change certain parameters in the model to be able to solve related problems. Learners are expected to see the relationships between objects in the model and expected to construct mathematical concepts through “mathematical abstraction”. This approach advocates creating specific models, activities and manipulatives, which is the main focus of this presentation, for every area of mathematics. Although there is a risk of rote-learning ready models without giving much thought, they might help learners gain problem solving skills which constitutes substantial part of mathematics curriculum. Before delving into manipulative models, mathematical abstraction process needs to be further explained because of its vital role in gaining conceptual understanding using those manipulatives.

MATHEMATICAL ABSTRACTION

Mathematical abstraction has long been on the agenda of educators (e.g., Dienes, 1963; Piaget, 1970) and this in turn amounted to a large literature on this issue. Ozmantar (2005), in his extensive literature review, investigates the issue of abstraction in mathematics education under two broad categories: cognitivist and socio-cultural views. Ozmantar (ibid.) extracts three main features associated with mathematical abstraction within the cognitivist tradition: (1) generalisation arising from the recognition of commonalities isolated in a large number of specific instances; (2) an ascent from lower concrete levels to higher levels of abstract thinking; and (3) a process of decontextualisation.

Piaget (1970), within this tradition, talks about three different types of abstraction depending on where one directs his/her focus of attention; empirical abstraction on objects, pseudo-empirical abstraction on properties and reflective abstraction on interrelationship among actions. Mathematical ideas are classified by deep structure rather than by visible appearance or known functions like everyday objects (Dienes, 1963). Dienes describes abstraction as "the extraction of what is common to a number of different situations" (ibid., p.57). In his view, abstraction is a process of discovering 'the same type of patterns' among different situations which embody the same concept, i.e., formation of an isomorphism, for example, by constructing rectangles from a given set of unit squares. Hence, a concrete experience in mathematics context is defined not by its physical or real-world characteristics but rather by how meaningful connections it could make with other mathematical ideas and situations. For instance, a student might create the meaning of the concept "four" by building a representation of the number and connecting it with either real or pictured blocks. Sfard (1991) argues that abstract mathematical notions can be conceived in two different ways; operationally as processes and structurally as objects. Learners firstly get familiar with mathematical concepts by using the processes or operations, manipulatives in our case, and their conception later is detached from the process and seen as a new object belonging to a particular category of concepts through reflection on these actions. Hence, it is very important to encourage learners to reflect on actions they make in order to be able to perceive mathematical processes as objects.

Regarding the socio-cultural view, Ozmantar (2005) suggests that accounts of abstraction in this tradition are greatly influenced by such authors as Lave (1988), Leont'ev (1978) and Vygotsky (1978) all of whom are concerned with the connection of learning and knowledge to, for example, the context of the learning, social interaction, personal histories, and to tools and artefacts available in a learning situation. In this respect, Lave (1988), for instance, analyses the performance of shoppers who perform the presented calculations virtually always correctly; nonetheless, these shoppers' success rate falls dramatically when they are presented with the same calculations in paper-and-pencil format. On the basis of this observation, Lave argues that the setting itself creates problems and structures its own solutions. In a similar vein, Brown et al. (1989) states that all knowledge is inextricably a product of the activity and situations in which they are produced and action is grounded in the concrete situations in which it occurs. Similarly, Resnick (1991, p.2) argues that "every cognitive act must be viewed as a specific response to a specific set of circumstances".

An important figure in the studies of abstraction in this tradition is Van Oers (2001) who describes abstract thinking as a process of contextualising an experience through the manipulation of physical materials and cycles of perceiving to discover new features and conceptual reframing. Noss and Hoyles (1996) asserts that context could affect one's cognition in many ways at varying degrees, for instance, depending on the tools and resources available at hand. Central to their argument is the presence of a structure of a particular situation, called webbing, that enables learners to make use of the previous constructions they have made and coining the term 'situated abstraction' when referring to how the webbing of a particular setting shapes the way in which the ideas are expressed.

Recent educational theories promote developing conceptual understanding rather than teaching procedures and memorizing facts and formula. Hiebert et al. (1986) states that conceptual knowledge can be regarded as a connected web of knowledge, a network in which the linking relationships between the individual facts and propositions are as prominent as the discrete pieces of information. The conceptual knowledge takes meaning with the explicit relationships in a context and cannot be explicitly represented as an isolated piece of information. Hence, conceptual knowledge grows by the construction of new knowledge, and the relationships between constructed concepts are strengthened when one practices with tasks involving those concepts. Therefore, it is very important to devise appropriate tasks to relay certain concepts and accomplish effective teaching. Meaningful educational activities and cognitive tools might improve students' active involvements in the teaching-learning process and encourage their reflections on the concepts and relations to be investigated.

When students perform tasks that they perceive as purposeful and authentic, they show greater interest in and accept more responsibility for their own learning and set their own personal meaningful goals (Jones et al., 1997; Savery & Duffy, 1995). Students also obtain significant gains in the educational contexts where they are challenged (Vygostky, 1978), and the construction of new mathematical concepts only occurs when a need arises (Dreyfus et al., 2001).

MANIPULATIVES AS A MODELING TOOL

Manipulative materials are concrete models that involve mathematical concepts, appealing to several senses including the socio-cultural needs that can be touched and moved around by the learners (Heddens, 2005). Manipulatives are physical objects, such as base-ten blocks, algebra tiles, Unifix Cubes, Cuisienaire rods, fraction pieces, pattern blocks and geometric solids that can make abstract ideas and symbols more meaningful and understandable to students. They are widely used in mathematics education. Furthermore, the usage of manipulatives in classrooms have long been recommended by educators (NCTM 1989, p. 17) and even mentioned in state legislations in Texas, Chapter 75, as "new concepts should be introduced with appropriate manipulatives at the elementary and secondary levels"(Peavler et al. 1987). While it is virtually impossible to demonstrate a mathematical concept directly by the help of manipulatives, it is likely for a learner to construct a concept or discover a mathematical relationship through appropriate use of manipulatives with an adequate task. It is suggested that manipulative materials can be used as an intermediary between the real world and the mathematical world (Lesh, 1979). Moreover, the usage of manipulative materials as concrete models thought to be more abstract than the actual situation but less abstract than the formal symbols (Post, 1981). Dienes (1961) emphasizes using manipulatives in order to provide a concrete referent for a concept, often at more than one level, instead of a referent for a given abstract idea or procedure. Concrete materials such as geometry rods, geoboard, isometric papers, symmetry mirrors etc. are supposed to help students construct geometric ideas. Using manipulatives benefits students across grade level, ability level, and topics which using manipulative makes sense for that topic (Driscoll, 1983; Sowell, 1989; Suydam, 1986). A simplistic design that enables easy manipulation should be chosen while creating manipulatives and motivational concerns should be addressed. Every student should be given an opportunity to play with manipulatives. Just a demonstration by a teacher is not sufficient to realize their full potential and not in line with the theoretical rationale of their usage since they are meaningful to the extent they involve interactive activities. Furthermore, manipulatives should be carefully chosen with the levels of intended audience and the realistic models, such as 1 stick for the digit 1 and 10 stick together as digit 10 for base blocks, should be used in order not to mislead learners by causing misconceptions.

Suydam and Higgins (1976) believe that lessons involving manipulative materials, if employed properly, will produce greater mathematical achievement than will lessons in which manipulative materials are not used. In fact, their meta-analysis of the studies using manipulatives verified them. They gave the following suggestions, in the same report, on good use of manipulatives:

1. Manipulative materials should be used frequently in a total mathematics program in a way consistent with the goals of the program.
2. Manipulative materials should be used in conjunction with other aids, including pictures, diagrams, textbooks, films, and similar materials.
3. Manipulative materials should be used in ways appropriate to mathematics content, and mathematics content should be adjusted to capitalize on manipulative approaches.
4. Manipulative materials should be used in conjunction with exploratory and inductive approaches.
5. The simplest possible materials should be employed.
6. Manipulative materials should be used with programs that encourage results to be recorded symbolically.

Heddens (2005) argue that using manipulative materials in teaching mathematics will help students learn:

- to relate real world situations to mathematics symbolism.
- to work together cooperatively in solving problems.
- to discuss mathematical ideas and concepts.
- to verbalize their mathematics thinking.
- to make presentations in front of a large group.
- that there are many different ways to solve problems.
- that mathematics problems can be symbolized in many different ways.
- that they can solve mathematics problems without just following teachers' directions.

Clements and McMillen (1996) proposed that using manipulatives does not always guarantee conceptual understanding: In one study, students not using manipulatives outperformed students using manipulatives on a test of transfer (Fennema, 1972). Furthermore, students sometimes used manipulatives in a rote manner (Hiebert and Wearne, 1992). Clements and McMillen (1996) claims that student often fail to link their action with manipulatives to describe the actions. Jackson (1979) identifies several common mistaken beliefs about manipulative materials including the facts that manipulatives do not necessarily simplify the learning of mathematical concepts, the more manipulatives used for a single concept-the better the concept is learned, and the manipulatives are more useful in the primary grades than in the intermediate and secondary grades, more useful with low-ability students than with high-ability students. In short, employing manipulatives in a class is not straightforward and good employment requires carefully defining the role of the teacher and the aims and the potentials of the tasks involved.

VIRTUAL MANIPULATIVES

A virtual manipulative is defined as "an interactive, web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge" (Moyer et al., 2002, p. 373). Visual representations of concepts and relations help learners to gain insight in mathematics. Virtual manipulatives enable as much engagement as physical manipulatives do since they are actual models of physical manipulatives mentioned above including Tangram and Geoboard (Dorwand & Heal, 1999). They may provide interactive environments where students could pose and solve their own problems to form connections between mathematical concepts and operations, and get immediate feedback about their actions that might lead them to reflect on their conceptualization. Although virtual manipulatives might simulate manipulatives in flesh, they are much more abstract since they do not allow hands-on activities. However, it is suggested that virtual manipulatives could be employed interchangeably with physical manipulatives in mathematics since manipulatives are not expected to make mathematical concepts “touchable” but to highlight the salient features of the concept to be covered. Hence, it is necessary to design specific math manipulatives focusing at different mathematical concepts. Virtual manipulatives might also provide further advantages over physical manipulatives by eliminating some of the constraints they impose on the task. Some computer manipulatives may be more beneficial than any physical manipulative. Artigue (2002) argues that mathematics education primarily does not aim to promote efficient mathematical practices with the help of available computational tools but rather concerned with the transmission of the bases of “mathematical culture”. Hence, efficient and successful use of virtual manipulatives is not self-evident and might require certain computational skills to be developed by a process of instrumentation. Furthermore, virtual manipulatives must be designed in a way to put focus on the mathematical concepts to be conveyed making their functionality as transparent as possible. Ozmantar (2005) argues that newly formed constructions are fragile entities and in need of consolidation. Hence, computer manipulatives could be used to reinforce the conceptual understanding. They could also be used to design extra-curricular activities since they are easily accessible both at home and the schools.

Any program having the following features can be thought as beneficial computer manipulative (Clements and McMillen, 1996, p.76). They

- ✓ have uncomplicated changing, repeating, and undoing actions;
- ✓ allow students to save configurations and sequences of actions;
- ✓ dynamically link different representations and maintain a tight connection between pictured objects and symbols;
- ✓ allow students and teachers to pose and solve their own problems; and
- ✓ allow students to develop increasing control of a flexible, extensible, mathematical tool. Such programs also serve many purposes and help form connections between mathematical ideas.

Selecting and using proper computer manipulative in learning environment should consider the following recommendations (Clements and McMillen, 1996, p.77):

- ✓ Use computer manipulatives for assessment as mirrors of students' thinking.
- ✓ Guide students to alter and reflect on their actions, always predicting and explaining.
- ✓ Create tasks that cause students to see conflicts or gaps in their thinking.
- ✓ Have students work cooperatively in pairs.
- ✓ If possible, use one computer and a large-screen display to focus and extend follow-up discussions with the class.
- ✓ Recognize that much information may have to be introduced before moving to work on computers, including the purpose of the software, ways to operate the hardware and software, mathematics content and problem solving strategies, and so on.
- ✓ Use extensible programs for long periods across topics when possible.

There are many funded projects in USA aiming to produce virtual manipulatives such as the national library of virtual manipulatives (NLVM) carried out by Utah State University (NLVM, 2005). Several java-based interactive mathematical manipulatives covering all areas of mathematics education at elementary and middle school levels have been designed by NLVM team. NLVM is designed in a way that manipulatives are presented both across the grade levels and mathematical strands (number sense and operations, measurement, geometry, algebra, and data analysis and probability). However, it might be argued some of the developed manipulatives lack the desired level of interactivity, usability and motivation since they employ predefined problem sets and provide limited interactivity. There are also special sites aiming specific subject areas of mathematics. Some examples are as follows: "The geometry applet" offers users a dynamic experience in three dimensional geometry (Joyce, 2005);"Algebra tiles" gives opportunities to users for investigating concepts and relations in algebra (Texas A,2005); "Base ten blocks activities" helps users gain insights about place value and arithmetic operations (Mankus,2005).There are many web sites hosting virtual manipulatives in mathematics in the form of applets or mathlets. The readers may visit those sites and may get insights about their functionality and reasoning (CTME, 2005).

CONCLUSION

The integration of technology into mathematics instruction requires students to be comfortable with new mathematical representations. Virtual manipulatives have been introduced in this article as viable computer applications both to get learners familiar with mathematical representations and to help them appreciate the meaningful applications of mathematics to solve real-world problems. Most manipulatives in mathematics simply implements the "learning with model" approach. However, educators also need to consider the possibility of designing manipulatives employing "learning to model" approach since full potential of any technological device could be achieved through its usage as a communication tool to model the concepts and relations at hand. The potential of virtual manipulatives for improving the quality of mathematics education is very promising since everyday new projects and web sites are developed for designing virtual manipulatives for some area of mathematics. Unfortunately, there seems to be no ongoing project in Turkey aiming to create computer-based mathematical manipulatives or learning tools. Hence, Turkish educational technologists should immediately start developing such projects in cooperation with mathematics educators. Although it is a good start to instigate campaigns to provide computer equipments and internet connections to every school in the country, policy makers should also focus on how these equipments will be employed to create learning environments providing thought-provoking activities. Direct translation of available virtual manipulatives into Turkish is not desirable and some cultural and contextual alterations to the design of the manipulatives might be required to meet the needs of Turkish audience since manipulatives could be regarded as a social medium.

REFERENCES

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning*, 7(3), pp 245-274.

Berry,J. & Houston K.(1995). Mathematical Modelling, Edward Arnold.

Brown, J. S., Collins, A. and Duguid, P. (1989), Situated cognition and the culture of learning. *Educational Researcher*, 18(1): 32-42.

Clements, D. H., & McMillen, S. (1996). Rethinking Concrete Manipulatives. *Teaching Children Mathematics*, 2(5), 270-279.

Collins, A., Brown, J.S., & Newman, S.E. (1989). Cognitive apprenticeship: Teaching the craft of reading, writing and mathematics. In L.B. Resnick (Ed.), *Knowing, learning, and instruction: Essays in honor of Robert Glaser* (pp. 453-493). Hillsdale, NJ: Lawrence Erlbaum.

Computing Technology for Math Excellence (2005). *Math manipulatives*. Accessed on September 2005 on site http://www.ct4me.net/math_manipulatives.htm

Dienes, Z. P. (1961). *Building Up Mathematics*. Hutchinson Ed. LTD London.

Dienes, Z. P. (1963). *An Experimental Study of Mathematics-Learning*. Hutchinson, London.

Dorward, J., & Heal, R.(1999). National Library of Virtual Manipulatives for Elementary and Middle Level Mathematics. *Proceedings of WebNet99 World Conference* on the WWW and Internet, pp. 1510-1512. Honolulu, Hawaii Association for the Advancement of Computing in Education.

Dreyfus, T., Hershkowitz, R., & Schwarz, B. (2001). Abstraction in context: the case of peer interaction. *Cognitive Science Quarterly*, 1(3): 307-368.

Driscoll, Mark J. (1983). *Research within Reach: Elementary School Mathematics and Reading*. St. Louis: CEMREL.

Duncan, K, Goldfinch, J. & Jackman,S (1996). Conference review of 7th International Conference on the Teaching of Mathematical Modelling and Applications. *International Reviews on Mathematical Education*, 28 (2), pp 67 - 69, 1996.

Fennema, E. H. (1972). Models and mathematics. *The Arithmetic Teacher*, 19, 635-640.

Heddens, J. W.(2005). *Improving Mathematics Teaching by Using Manipulative*s. Accessed on September 2005 on site http://www.fed.cuhk.edu.hk/~fllee/mathfor/edumath/9706/13hedden.html

Hiebert, J. (1986). *Conceptual and Procedural Knowledge: the case of Mathematics*. Lawrence Earlbaum associates, Publishers, London.

Hiebert, J., & Wearne, D. (1992). Links between teaching and learning place value with understanding in first grade. *Journal for research in Mathematics Education*, 23, 98-122.

Jackson,. R.(1979). Hands--on Math: Misconceptions and Abuses. *Learning*, 7: 76-78.

Jones, B. F., Rasmussen, C. M. & Moffitt, M.C. (1977). *Real life problem solving: A collaborative approach to interdisciplinary learning.* Washington, DC: American Psychological Association.

Joyce, D.E(2005). *The Geometry Applet* . Accessed on September 2005 on site http://aleph0.clarku.edu/~djoyce/java/Geometry/Geometry.html

Koedinger, K. R., Anderson, J. R., Hadley, W. H. & Mark, M. A. (1997). Intelligent Tutoring Goes to School in the Big City. *Journal of Artificial Intelligence in Education*, 8 (1), 30-43.

Leont'ev, A. N. (1978). *Activity, Consciousness and Personality*. Prentice Hall, Englewood Cliffs.

Lesh, R. A. (1979). *Applied Problem Solving in Early Mathematics Learning*. Unpublished working paper, Northwestern University.

Lave, J. (1988). Cognition in Practice: Mind, Mathematics and Culture in Everyday Life. Cambridge University Press, Cambridge.

Mankus, M.L. (2005). *Base Ten Block Activities*. Accessed on September 2005 on site http://www.frontiernet.net/~mmankus/whole/base10/baseten.htm

Moyer, P. S., Bolyard, J.J., & Spikell, M.A. (2002). What are virtual manipulatives? *Teaching Children Mathematics*, 8(6), 372-377.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA.

Nattional Library of Virtual Manipulatives. (2005). *Virtual Manipulatives, Utah State University.* Accessed on September 2005 on site http://nlvm.usu.edu/en/nav/index.html

Norman, D. A. (1993). *Things that make us smart: Defending human attributes in the age of the machine*. Reading, MA: Addison-Wesley Publishing Co.

Noss, R. & Hoyles, C. (1996). *Windows on Mathematical Meanings: Learning Cultures and Computers*. Kluwer, Dordrecht.

Ozmantar, M F (2005). *An investigation of the formation of mathematical abstractions through scaffolding*, Unpublished PhD Thesis, Univeristy of Leeds.

Peavler, C., DeValcourt, R., Montalto, & B., Hopkins, B. (1987). The mathematics program: An overview and explanation. *Focus on Learning Problems in Mathematic*s, 9, 39-50.

Piaget, J. (1970). Genetic Epistemology. W. W. Norton, New York.

Post, T. (1981). *The Role of Manipulative Materials in the Learning of Mathematical Concepts. In Selected Issues in Mathematics Education*. Berkeley, CA: National Society for the Study of Education and National Council of Teachers of Mathematics, McCutchan Publishing Corporation.

Resnick, L. (1991). Shared cognition: thinking as a social practice. In L. Resnick, J. Levine and S. Teasley (eds.), *Perspectives on Socially Shared Cognition*, (pp.1-20). American Psychological Association, Washington, DC.

Salomon, G. , Perkins, D.N., & Globerson, T. (1991). Partners in cognition: Extending human intelligence with intelligent technologies. *Educational Researcher*, 20(3), 2-9.

Savery, J. R., & Duffy, T. M. (1995). Problem-Based Learning: An Instructional Model and its Constructionist Framework. *Educational Technology*, Sept - Oct, 31 - 38.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies In Mathematics*, 22: 1-36.

Simon, H. A.(1981). *The Sciences of the Artificial*, The MIT Press, Cambridge, MA.

Skinner, B. F. (1954). The science of learning and the art of teaching. *Harvard Educational Review*, 24(2), 86-97.

Science and Mathematics Initiative for Learning Enhancement .(2005). *Algebra Tiles*. Accessed on September 2005 on site http://www.coe.tamu.edu/~strader/Mathematics/Algebra/AlgebraTiles/

Sowell, Evelyn J. (1989). Effects of Manipulative Materials in Mathematics Instruction. *Journal for Research in Mathematics Education*, 20: 498-505.

Suydam, M. N., & Higgins, J. L.(1976). Review and Synthesis of Studies of Activity-Based Approaches to Mathematics Teaching. Final Report, NIE Contract No. 400-75-0063.

Suydam, M. N. (1986). Research Report: Manipulative Materials and Achievement. *Arithmetic Teacher*, 33:10, 32.

van Oers, B. (2001), 'Contextualisation for abstraction'. Cognitive Science Quarterly, 1(3): 279-305

Vygotsky, L. S. (1978). *Mind in society. The development of higher psychological processes*. Cambridge, MA: Harvard University Press.