TEACHING THE *DIAGONALIZATION CONCEPT* IN LINEAR ALGEBRA WITH TECHNOLOGY: A CASE STUDY AT GALATASARAY UNIVERSITY

Assist. Prof Dr. Ayşegül YILDIZ ULUS
Galatasaray University, Department of Mathematics
Çırağan cad. No:36 Ortaköy Istanbul, Turkey
ayildizulus@gmail.com

ABSTRACT
This paper examines experimental and algorithmic contributions of advanced calculators (graphing and computer algebra system, CAS) in teaching the concept of *diagonalization*, one of the key topics in Linear Algebra courses taught at the undergraduate level. Specifically, the proposed hypothesis of this study is to assess the effective utilization of advanced calculators as an assistance tool in furthering the conceptual understanding of diagonalization. . To this end, using some instructional materials, we present a new dimension of teaching as a combination of a theoretical experimentation and algorithmic approach. The primary findings indicate that advanced calculators provide carefully designed tasks in favor of the acquisition of mathematical knowledge while facilitating the students' perception of the concept, both valuable tools in the instructional delivery of *diagonalization* as a concept.
Keywords*:* Linear Algebra, diagonalization, ICT, advanced calculators, theoretical experimentation, algorithmic approach.

1. INTRODUCTION

Linear Algebra courses occupy a dominant position in the overall undergraduate mathematics curriculum. The concepts taught within these courses lend themselves to the understanding of more advanced mathematics courses. This further extends to the maximization of these theories as transferable knowledge to courses in engineering, economics, physical and chemical sciences, and statistics. Primarily, Linear Algebra courses consist the study of matrices, systems of linear equations, vector spaces, inner products, orthogonality and the theory of *diagonalization*. As other areas of mathematics at the collegiate level, Linear Algebra offers various tools with a broader scope of applications. Cowen (1997), for example, pointed out that Linear Algebra is a course that is full of ideas, including materials that are rewarding to learn and teach, and is a subject where both student and teacher can be challenged to their best performance. Reforms in teaching of Linear Algebra have been moving forward over the last two decades; in 1993 the Linear Algebra Curriculum Study Group (LACSG) in the USA published a set of recommendations for the first course in Linear Algebra (Carlson, Johnson, Lay, & Porter, 1993). These recommendations stimulated the research on how to teach Linear Algebra and led to an extensive literature in recent years. For instance, the book of *Resources for Teaching Linear Algebra* published by Mathematical Association of America (MAA) (Carlson et al., 1997) has always been one of the essential reference sources on this topic.

It is important to note that the use of Information and Communication Technologies (ICT) in mathematics at the collegiate level has already been widespread. Several researchers have indicated that the use of ICT impacts both students' attitudes and academic achievements. Moreover, relative to traditional methods, it has also been emphasized that the use of technological materials helps in improving students' mathematical skills by directing them to focus on understanding of the concepts. In light of such evidence, it is essential for the researchers studying Linear Algebra to have a better understanding of how to incorporate ICT effectively in teaching and learning.

The availability of several ICT tools has offered new opportunities, specifically their powerful symbolic, graphic and programming capabilities in various concepts and topics related with Linear Algebra. Nevertheless, taking advantage of ICT requires reconsidering teaching and learning of many topics in the Linear Algebra curriculum. Day and Kalman (1999) have mentioned that the important question for Linear Algebra teachers to ponder is whether, when, and how to use the technology while teaching. In that regard, there have been several different views and experiences of using technology in teaching and learning. On one hand, some teachers and researchers reject using technology in teaching with the fear of turning their students into unthinking button-pushers. On the other hand, several instructors encourage the extensive use of ICT tools in order to improve both the learning and teaching experiences (the software programs like MatLab, Octave, Maple, Sage, Mathematica, Mathwright have been used extensively in recent years while some teachers and researchers still prefer advanced calculators to these software). Similarly, while some teachers assign computer projects to be done outside of the classroom, others utilize computer demonstrations and examples in the classroom to enrich their lectures.

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Cowen (1997) claimed that, besides playing a computational assistant role, ICT tools provide motivation in understanding and learning the theory, and reinforcing the relevant concepts. In Linear Algebra, unlike many other undergraduate mathematics courses, theory plays an essential role in computations. Thus, the impact of ICT use in general and in some hand-held personal technologies, namely advanced calculators on the student learning experience has been a popular research area in mathematics education in many countries, including Turkey in recent years (Ersoy, 2003a, 2003b, 2005). After recalling various changes that have taken place in the last two decades, Ersoy (2005) emphasized that the use of advanced calculators has been a reform movement in the process of teaching and learning mathematics. In this context, he further explained the concept of calculator-supported/assisted mathematics instruction and summarized briefly the innovations in various countries over the last two decades.

This paper is a part of the report of a project supported by Galatasaray University, Istanbul, Turkey. The Project has two major goals: (1) to investigate how to incorporate ICT into undergraduate level mathematics courses; and (2) to develop a series of new instructional materials on several topics within undergraduate level mathematics. Specifically, the present paper deals with the concept of diagonalization, which has a great importance among many interesting topics in Linear Algebra. It is built on the results from the study of matrices and vector spaces and is applicable in many areas such as algebraic geometry, differential equations, and physics. By utilizing this concept, it is possible to canonicalize a system of equations in a simpler form. With a square *nxn* matrix, one has a system of *nxn* equations, but if the matrix is diagonalizable, the number of equations can be reduced to *n*. However, as might be expected, the diagonalization process of a square matrix requires some complicated matrix calculations like finding the determinant of a matrix and the solution spaces of homogeneous systems by reducing the size of matrices. The steps of this process are difficult to follow and there is a great risk of making mistakes in the calculations, especially if the size of the matrix gets large. After the matrix calculations, the critical part of the process is to decide whether the matrix is diagonalizable or not. Thus, it is interesting to observe the experimental and the algorithmic contributions of technology in the concept of diagonalization.

In the following section, various observations, insights and experiences are presented about the introduction, implementation, and integration of ICT in undergraduate level mathematics education. While the third section presents the methodology, the fourth section describes the instructional materials designed for this particular concept in the form of worksheets that focus on motivating examples, the algorithmic process, and the programs. The fifth section is the research part of the study where students were interviewed with a set of questions in order to gather students' views on the designed instructional materials and the benefits of ICT in Linear Algebra courses I and II. The collected data is analyzed by the content analysis method. Conclusions and suggestions are presented in the final section.

2. THEORETICAL FRAMEWORK: BACKGROUND INFORMATON

The teaching and learning of Linear Algebra concepts and their applications have a short but an interesting story. Reforms in the curriculum and the use of the appropriate technology for this course have been a focus of research within the society of mathematics education in several countries. Therefore, it is valuable to give a short review of these developments and pay attention to certain issues on this topic.

A General Overview on Learning Mathematics: In the field of mathematics education, some models have been developed to determine how students learn mathematics. Conceptual and procedural learning have been addressed by mathematics educators and others. For example, following the well-known Piaget's theory, Gray and Tall (1994) introduced the idea of a *procept* and Dubinsky and McDonald (2001) introduced the famous *APOS* (Action-Process-Object-Schema) theory which has been interpreted by some researchers as an extension of Piagetian theories for the higher level of abstract mathematics. Based on the *APOS* theory, there has been a set of new developments in teaching approaches and strategies in the field since the given theory provides a variety of ways about how to assimilate and learn abstract mathematics.

Given these developments, Laborde (2007) claimed that various systems of representation in mathematics that have been built over time affect how we do mathematics. Mathematical objects are only indirectly accessible by some representation forms such as diagrams, schemas, figures, formulas, tables, graphics, and algorithms, among others. Moreover, mathematical activities require some manipulations and operations on these different forms of representations (D'Amore, 2003; Duval, 2000). It is supported by the related existing literature that teaching with the use of multiple representation forms provides a more complete understanding of the mathematical concepts relative to traditional ways of teaching. At this point, the role of integrating ICT into the teaching of mathematics is crucial since with these available technologies it becomes feasible to construct the bridge between different representations in an easier and a more compact manner.

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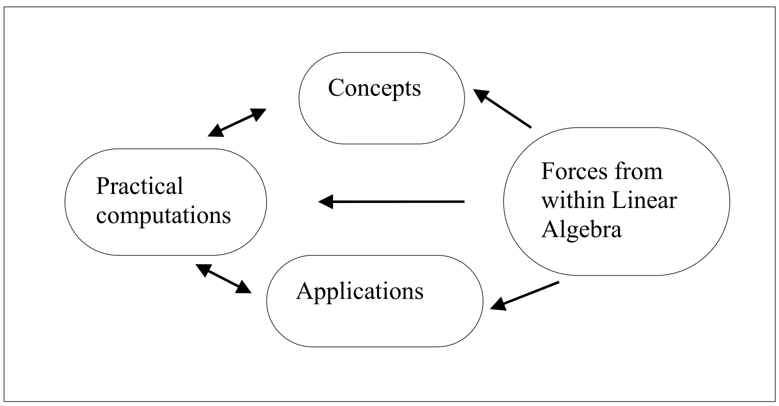
Teaching and Learning Linear Algebra: Linear Algebra is an abstract area of mathematics taught only at the graduate level up to the 1960s in almost all countries. Since then, it has started appearing at the undergraduate level and its applications in various fields have become quite common. In 1990, the LACSG in the USA produced a recommended core curriculum for the first Linear Algebra course at the undergraduate level. Then, the LACSG published a report in 1993 about who needs Linear Algebra, why it is needed, and what might be the scope of a first course. For a short review of published literature on how students learn Linear Algebra, Harel (1998) seems to be the first to mention some suggestions for Linear Algebra teachers. Carlson (1993) (who was in fact a member of LACSG) added some important points to the recommendations of the LACSG by specifying the major topics in Linear Algebra that are problematic to students. Dubinsky (1997) offered an alternative project about learning Linear Algebra that can be seen as an extension to the *Calculus Reform.* His approach makes use of students programming computers, cooperative learning, and other alternatives to lecturing. Day and Kalman (2001) presented some issues and resources on the teaching of Linear Algebra that contain goals of instruction, materials to cover, methods of instruction, instructional technology, levels of abstractness and rigorousness, applications, student diversity, connections with other courses, and many others. Finally, Uhlig (2003) developed a different approach which he called *balanced approach* for the teaching of Linear Algebra. According to this approach, a successful comprehensive teaching method can be developed by finding a balance between the parts of the following diagram (see Schema 1) accompanied by interlocking pedagogical principles: openness of teaching and exploration, concreteness (hands-on computations), usefulness (applications) and creativity.



Schema 1. A Comprehensive Approach to Teaching Linear Algebra (Uhlig, 2003)

Technology in Teaching/Learning Mathematics and Linear Algebra: Traditional methods of teaching and learning are no longer adequate to meet the demands of higher education. Many different types of technology can be used to support and improve both learning and teaching experiences in higher education. For example, the communication skills can be promoted via e-mail exchanges and e-groups; the organizational skills can be improved via database and spreadsheet programs; numerical and symbolic computations and graphing of functions can be made effortlessly by using hand-held technologies such as advanced calculators, and finally the understanding of mathematical concepts can be better promoted with the use of some modeling software programs. Thus, it is important to consider how these technologies differ and sort out which of the characteristics make them important as vehicles for the teaching and learning of mathematics (Burrill et al., 2002; Ersoy, 2003a, 2003b; National Council of Teachers of Mathematics [NCTM], 2000).

Today, there are two main technological tools that are used in teaching and learning of Linear Algebra: the hand-held tools (such as advanced calculators) and the mathematical software programs. They help students perform complex matrix computations instantaneously and effortlessly. Thus, instead of spending lots of effort and time on heavy computations, students are able to concentrate on the main questions about the nature of the operations. This is the ultimate goal of many instructors involving technology in their teaching. While some instructors feel that doing some of the matrix multiplications by hand provides insight about why results appear as they do, many others believe that the ability to rapidly investigate a large number of examples makes a valuable contribution to students' understanding (Day & Kalman, 1999).

Hand-held technologies, such as TI-84 plus, TI Voyage or nSpice and *Casio-ClassPad 330*, are endowed with very powerful software as well as the programming options, allowing the undergraduate students to solve advanced problems that were previously identified as being highly difficult. Similar to hand-held versions, emulator software also allows teaching and learning in many different ways. Since there is no need for computer

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labs, students can benefit from these tools without a restraint of time and place: in school, at home, even in the examinations if the teachers permit. On the other hand, by utilizing this hand-held software, teachers may enrich their lectures by allowing exploration and rapid hands-on computations. As a result, this leads to an efficient way of using technology for students and teachers in or out of school.

The Project on Teaching Linear Algebra at Galatasaray University: In 2008, a mobile mathematics laboratory was established in the Department of Mathematics at Galatasaray University. By means of the mobile laboratory, the main plan was to teach Calculus and Linear Algebra courses with the use of appropriate technology, carry out some research on how students learn basic concepts of the courses, enhance basic skills, apply the theoretical knowledge, and design new instructional materials. The ultimate aim of the project was the development and improvement of students' mathematical abilities, understanding, and appreciation of the use of advanced calculators.

Laborde (2007) claimed that the effective use of technology can be provided when three main conditions are met: the choice of the proper software, the clearly outlined/designed tasks given to the students, and the properly identified role of the teacher. Consistent with Laborde (2007), the present project has given emphasis to these conditions. The outcomes for learning generally depend not only on the instructional tasks but also on the software design. Thus, the interface of calculators that satisfy both the criteria of *utility* (related to the learning potential) and *usability* (being user-friendly) is a critical element of this design. Properly designed technology-based tasks provide vast possibilities of actions in numbers and in the nature of operations, and also enable students to conjecture about certain mathematical facts. They also offer students multiple representations of a concept and open up various possibilities to develop control strategies. How teachers use technology themselves and how they guide the technology-based tasks seem to be a critical component of effectiveness. While performing the actions of the task, the presence of the teacher is important in assisting the students, explaining what the calculator does, and what it means mathematically. This way, teachers can connect tasks with more theoretical perspectives rather than simply focusing on computations and hence contribute to the internalization of the students' knowledge.

3. METHODOLOGY

Rationale: In contrast to the other areas of mathematics education, current research about the use of the advanced calculators in teaching and learning of Linear Algebra is still in its infancy. Our ultimate aim in this case study was to shed light on the impact of introducing and integrating advanced calculators into the teaching environment and setting up a mobile mathematics laboratory on the teaching/learning experiences in Linear Algebra. However, as the review of research at the international level (Lagrange, Artigue, Laborde, & Trouche, 2003) pointed out that there are many difficulties one expects to face while integrating computational tools into educational institutions. These difficulties stem from various problems, such as the large number of students in classes, hesitation and unwillingness of the instructors to utilize ICT tools in teaching, and the problem of curriculum changes.

Goals: This study had two major goals: (1) to investigate how to incorporate ICT into undergraduate level mathematics; and (2) to develop a series of new instructional materials on several topics of undergraduate level mathematics.

Research Problems: The research problems/questions in the study were: (a) How an appropriate use of ICT can be organized taking into account of the problems cited above and (b) How can we take full advantage of such powerful tools in order to develop a deep conceptual understanding of diagonalization in Linear Algebra? One of the original points of this study comes from the introduction of a new dimension of teaching which is a combination of *theoretical experimentation* (this notion is due to Borwein, Borwein, Girgensohn, and Parnes, 1996) and an algorithmic approach. The second important aspect of this research is the fact that this is the first experimental study on teaching a Linear Algebra course with the integration of advanced calculators.

In the application, the advanced calculators (namely *Casio-ClassPad 330*) with programming options were utilized. The theoretical experimentation included structuring a domain to formulate a hypothesis, deriving motivating examples to solve with a calculator, and interpreting the calculator's results. The algorithmic dimension included constructing algorithms and writing programs. Lagrange (2005) mentioned that the algorithmic dimension by using calculator needs a change of teaching regarding the utilization of the programming options of advanced calculators. However, there is a little didactical research offering help about this change. Nevertheless, an algorithmic way of thinking and sufficient programming skills led to interesting applications.

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Method: A qualitative case study method was used in order to conduct this study. First of all, as clearly indicated in Yin (1994), this approach is appropriate to our above mentioned research problems. Second, consistent with the case study approach, a set of procedures such as designing teaching materials, observing while teaching and collecting data, analyzing the information and reporting the results were followed in the study. A questionnaire was applied to gather the students' views on the designed instructional materials and the benefits of ICT (the advanced calculator) in Linear Algebra courses I and II. The content analyses were conducted on students' written answers as well as on group interviews. Students' success was assessed by several homework tasks and specific questions in midterm and final exams. Finally, this study, as its name suggests, was a case study and thus it was not aimed to draw general conclusions within the outlined context.

4. TEACHING PRACTICE AND DEVELOPMENT OF INSTRUCTIONAL MATERIALS

Teaching Practice: First, the teaching practices were carried out in the second year classes in the Department of Mathematics at Galatasaray University. In the academic years 2008-09, there were only 6 students, all equipped with an advanced calculator (*Casio-ClassPad 330*). They were the first group of students whose curriculum was adapted to teaching and learning with technology. In the following academic years 2009-10, 2010-11 the applications were replicated *without any change*, this time, with 10 and 19 students respectively. Therefore, our data and results considered 35 students in total. It is worth underlining the fact that the students were well-qualified in programming since they started to learn some computational languages such as ISETL and C++ in the first year of their education and subsequently they took Algorithms and Advanced Programming Courses in the second year.

First, students were motivated in the classroom with the use of exercises and then the theory of the diagonalization was taught and some related problems were solved by using paper-and-pencil skills. In order to carry out the pre-planned activities, some instructional materials were designed in the form of worksheets involving exploration, investigation and challenging problems on which they used advanced calculators. Then, an algorithm was constructed with the help of the flow charts and finally a program was written which helped decide the diagonalizability of a matrix. In the recitation hours, these worksheets were given to the students and observations were noted.

Use of ICT in Teaching the Process of D*iagonalization*

First, by defining the diagonal and other related matrices, the idea behind the concept was clearly outlined to the students in the classroom. With the help of its similarity to a diagonal matrix, some problems were set out such as solving a system of equations which evoked the idea of being able to reduce the number of equations. Second, the theory of the diagonalization was taught by giving the definitions and the theorems with their proofs. Here we revisited the following: an nxn matrix A is diagonalizable if it has n linearly independent eigenvectors and a square matrix A is diagonalizable over R if there is an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

Then the process of *diagonalization* was introduced:

Step 1: Find the characteristic polynomial of the matrix A, which is the determinant of the matrix A-xI.

Step 2: Find the eigenvalues x_i of A which serve as the roots of the characteristic polynomial.

- If there are n distinct eigenvalues x_1 ... x_n, A is diagonalizable. Then there exists a diagonal matrix D whose diagonal elements are these eigenvalues such that $P^{-1}AP = D$. Then, we continue to *Step 3* in order to obtain P.
- If all eigenvalues are not distinct, continue to *Step 3.*

Step 3: For each eigenvalue x_i, for $i=1...r$ where $r<n$, determine the corresponding eigenspaces. In other words, solve the homogeneous system $(A- x_iI)\, u=0$ and find a basis for its solution space. In this step, one has to reduce the matrix $A- x_iI$ and solve the equivalent system. Choosing a base in every eigenspace E_{xi} and the union of these bases constitute a base of eigenvectors (a non-zero column vector u such that $Au= x_i u$).

Step 4: Using the results of *Step 3*, determine whether A is diagonalizable or not. If one can find n linearly independent eigenvectors $u_1...u_n$, then A is diagonalizable and P is the passage matrix to this new base of eigenvectors, and there exists a diagonal matrix D such that $P^{-1}AP = D$. Otherwise, it means dim E_{x1}+…. + dim $E_{xr} \neq n$ and A is not diagonalizable. This last result is due to a corollary of the theorem which states a square nxn matrix is diagonalizable if and only if dim E_{x1}+…. + dim $E_{xr}= n$ for some distinct eigenvalues x_1... x_r with $r<n$ or $r=n$.

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Finally, some problems were solved in the recitation hours by using paper-and-pencil skills. Then, students were given a worksheet including examples on which they used advanced calculators in order see how easy it was to handle the calculations by the use of calculators relative to the use of paper-pen-pencil skills only. This was also an opportunity for them to become familiar with the commands of the calculators, explore and conjecture themselves about several interesting mathematical facts. For example, one of the questions in the given worksheet was the computation of the matrix power or matrix exponential. After computations on several matrices, they all conjectured correctly and 6 of the students (6/35, ~ %17) could prove it by using the concept of diagonalization.

Then, the students were asked in a quiz to decide whether the given two *3x3* matrices are diagonalizable or not and find the matrix *P* for the diagonalizable matrix. They were allowed to use the calculators in the first three steps: In *Step 1* the command "det(*A-xI*)" was used to find the characteristic polynomial *q(x)*, in *Step 2* "factor (*q(x)*)" or "solve(*q(x)*=0,*x*)" was used to find the roots of the characteristic polynomial and in *Step 3* "rref($A-x_iI$)" was used to reduce the corresponding matrix $A-x_iI$. Seven (7/35, 20%) of the students, who preferred to find the determinant by paper-and-pencil method in the first step, could not evaluate it and could not continue to the next steps. The rest of students who preferred to use the calculators could handle these calculations without mistakes. In *Step 4*, for the diagonalizable matrix, they were able to give the correct answer and using the basis vectors of the eigenspaces they correctly constructed the matrix *P*. However, for the matrix which is not diagonalizable, they all tried to use the main theorem but none of them could correctly state the arguments.

Subsequently, in the mid-term examination they were given an exercise in which there were three *3x3* matrices, and they were asked to decide whether the matrices were diagonalizable or not, via using the experimental approach and the theoretical experimentation approach.

Experimental Approach: Two of the matrices (the matrices *A* and *B* below in Figure 1) are diagonalizable and the other one (the matrix *C* below in Figure 2) is not. By the experimental approach, they were allowed to pass directly the first three steps and obtain the eigenvalues and eigenvectors and conclude about the diagonalizability of the matrices. For the matrix *B* which is diagonalizable and the matrix *C* which is not diagonalizable, in the last step, nineteen (19/35, ~ 54%) of the students utilized the corollary of the theorem (this result is also underlined in the redaction of the program) and stated their decisions correctly. (See Figure 1 and Figure 2).

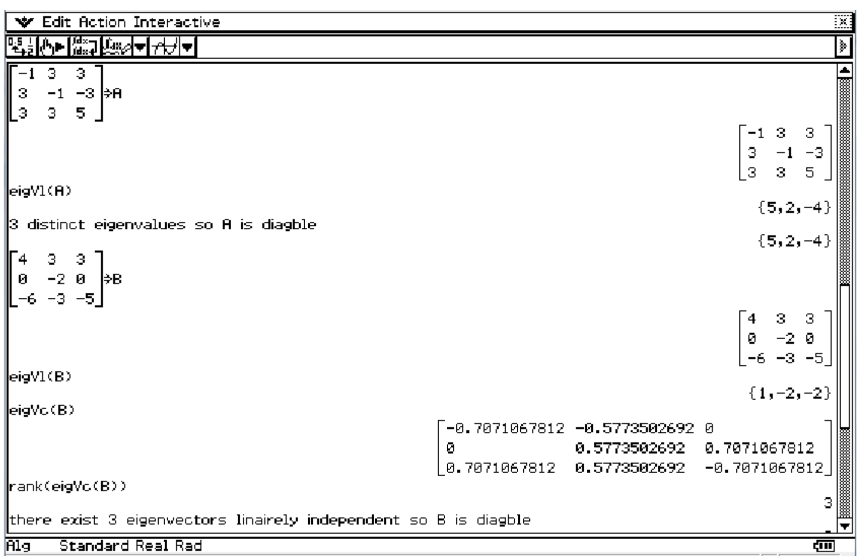

Figure 1. A representation of "diagonalizability of the matrices A and B" by the matrix computations of *Casio-Classpad 330*.

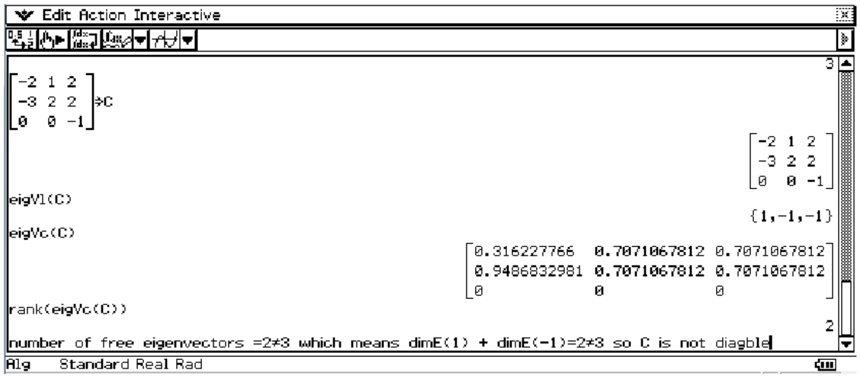

Figure 2. A representation of "non-diagonalizability of the matrix C" by the matrix computations of *Casio-Classpad 330*.

Theoretical Experimentation Approach: By the theoretical experimentation approach, while all of the students could follow the steps separately and state easily the diagonalizability of the matrices *A* and *B*, fifteen of them (~ 43%) could correctly find the non-diagonalizability of the matrix *C*. This was the control action of the experimental approach which provided a flow chart and then an algorithm. (See Figure 3-5 below).

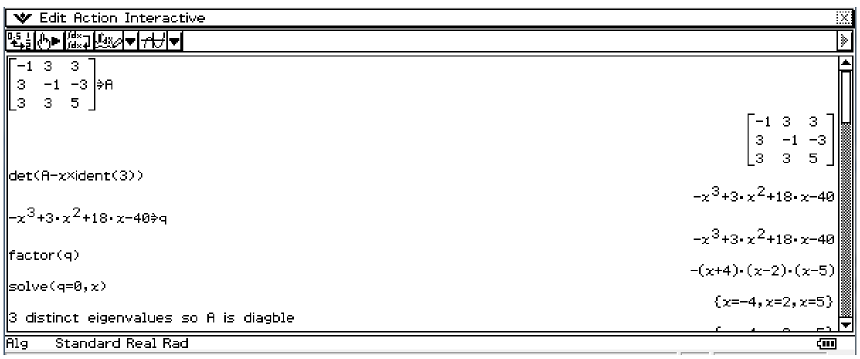

Figure 3. A representation of "diagonalizability of the matrix A" by the theoretical experimentation approach on *Casio-Classpad 330*.

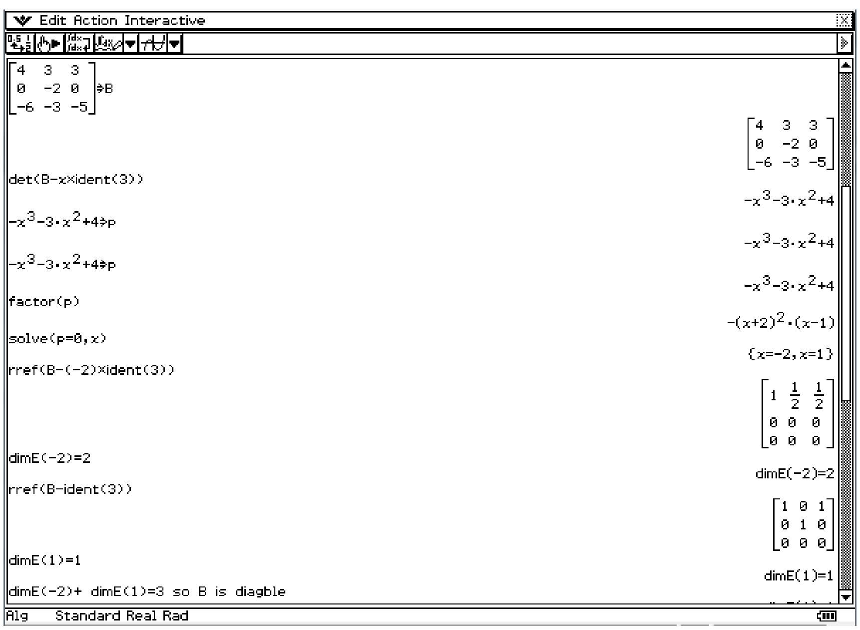

Figure 4. A representation of "diagonalizability of the matrix B" by the theoretical experimentation on *Casio-Classpad 330*.

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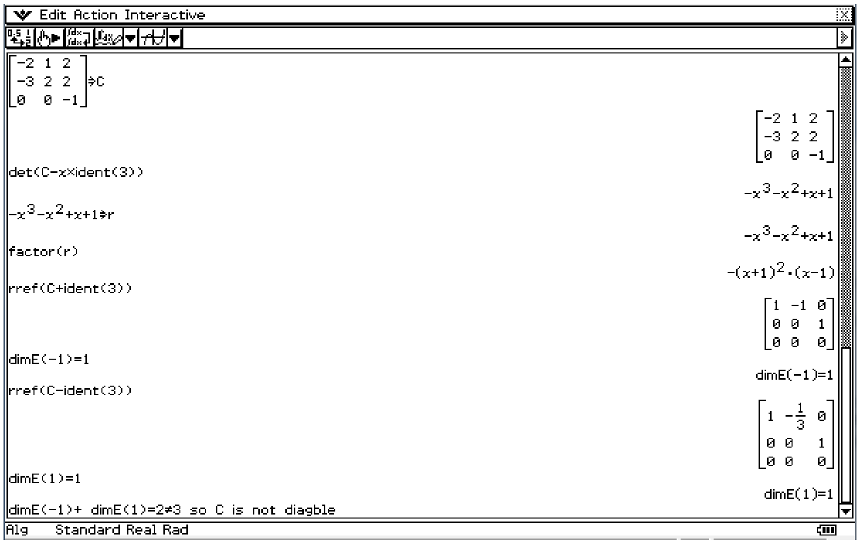

Figure 5. A representation of "non-diagonalizability of the matrix C" by the theoretical experimentation on *Casio-Classpad 330*.

Algorithmic Approach: After the midterm, a discussion was organized to explore whether a relevant program could be written which would tell us directly if a matrix is diagonalizable immediately after the inputting of a matrix. The idea appeared to be exciting to the students and they all agreed that this would not be difficult since the process of diagonalization was given like a flow chart. Moreover, fortunately, the programming language of advanced calculators is not complicated. What remains was constructing an algorithm. To this end, a program was written interactively together with the students (see Figure 6). As the major objective was to improve algorithmic thinking capability of the students, the steps which cover all of the possible cases were considered following a theoretical experimentation. In the last step of the process, this program utilized one of the corollaries of the main theorem which tells that a matrix A is diagonalizable if and only if the sum of the dimensions of its eigenspaces is equal to its dimension.

Following this application, it was observed in the final exam that none of the students encountered a problem in the exercises in which this concept was required. It is worth noting that, the advanced calculators were not authorized in the final exam. Hence, we were able to conclude that previously designed instructional materials with experimental, theoretical experimental approaches and algorithmic options of advanced calculators were successful in the sense that they contributed to the internalization of the knowledge.

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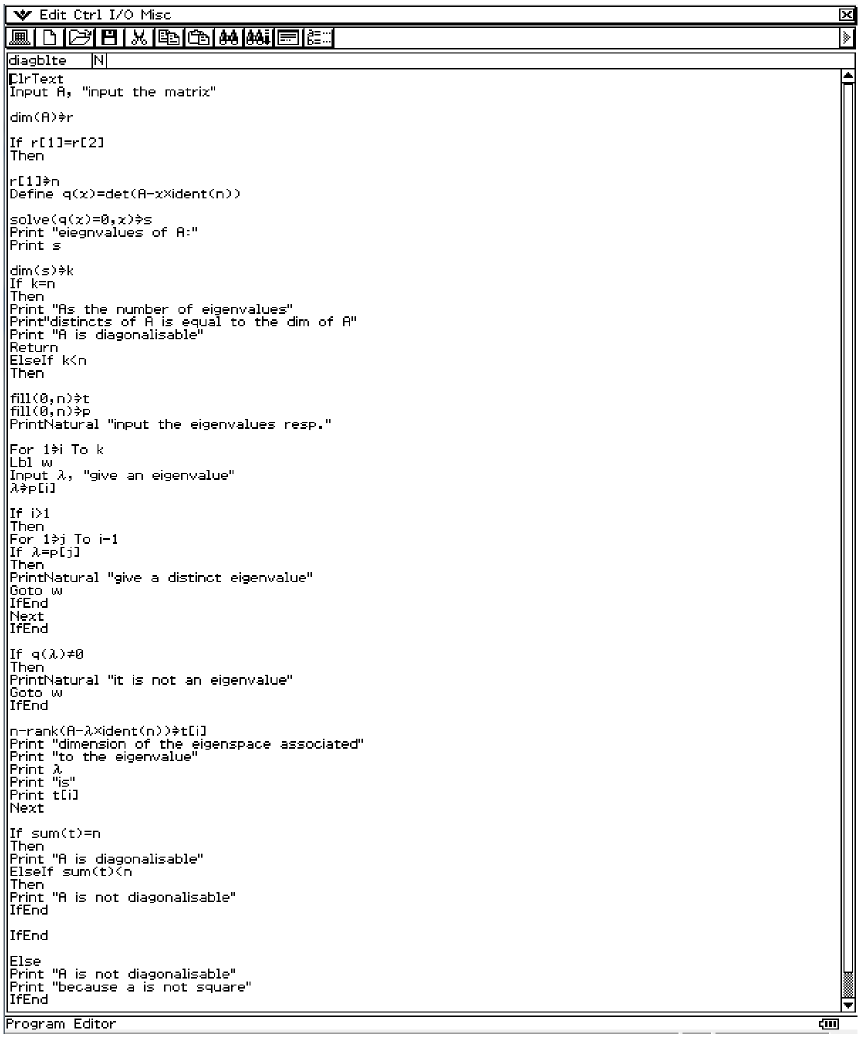


```
ClrText
Input A, "input the matrix"

dim(A)⇒r

If r[1]=r[2]
Then

r[1]⇒n
Define q(x)=det(A-x×ident(n))

solve(q(x)=0,x)⇒s
Print "eiegnvalues of A:"
Print s

dim(s)⇒k
If k=n
Then
Print "As the number of eigenvalues"
Print"distincts of A is equal to the dim of A"
Print "A is diagonalisable"
Return
ElseIf k<n
Then

fill(0,n)⇒t
fill(0,n)⇒p
PrintNatural "input the eigenvalues resp."

For 1⇒i To k
Lbl w
Input λ, "give an eigenvalue"
λ⇒p[i]

If i>1
Then
For 1⇒j To i-1
If λ=p[j]
Then
PrintNatural "give a distinct eigenvalue"
Goto w
IfEnd
Next
IfEnd

If q(λ)≠0
Then
PrintNatural "it is not an eigenvalue"
Goto w
IfEnd

n-rank(A-λ×ident(n))⇒t[i]
Print "dimension of the eigenspace associated"
Print "to the eigenvalue"
Print λ
Print "is"
Print t[i]
Next

If sum(t)=n
Then
Print "A is diagonalisable"
ElseIf sum(t)<n
Then
Print "A is not diagonalisable"
IfEnd

IfEnd

Else
Print "A is not diagonalisable"
Print "because a is not square"
IfEnd
```

Figure 6. A Program which tests the digonalizabilty of a square matrix written on *Casio-Classpad 330*

5. FINDINGS AND RESULTS

Here, it is necessary to note that while the small number of students in this study may possibly pose problems about the reliability of the results, it also has numerous advantages. For instance, it allowed us to equip all the students with the necessary devices and also directly observe the contribution of the ICT tools.

A student group interview was conducted along four main themes: (1) using advanced calculators; (2) teachers and teaching; (3) students' learning and (4) advice. As a general questionnaire, the students' opinions were obtained about teaching with technology in Linear Algebra. Finally, students reflected on their impressions, personal opinions, and suggestions concerning the concept of diagonalization and the use of calculators on a topic questionnaire. First of all, teaching with technology was appreciated by all of the students involved. Advanced calculators, with their hand-held and practical use, enriched the lectures, especially the recitation hours along several dimensions. They investigated, explored, compared their results and discussed the concepts. Constructing an algorithm and a program summarized the mechanism underlying the relations of the reason-result. Of course, as might be expected, the students who had a special interest in programming and who did their homework regularly succeeded relatively better.

Table 1 gives an overall content assessment of the application. Findings were then classified in two main dimensions: experimental and algorithmic (see Table 2 and Table 3 for the details).

In Table 1; 5: I completely agree, 4: I agree, 3: I am undecided, 2: I don't agree, 1: I completely disagree. The word ICT is preferred to use in tables, but it is clear that ICT means advanced calculators in this context (*Casio-Classpad 330*).

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Table 1. General opinions about the content of the application of "teaching with ICT".

General Assessment	**5**	**4**	**3**	**2**	**1**
	(f)	(f)	(f)	(f)	(f)
ICT helps me to understand more deeply the concepts in Linear Algebra I and II	19	11	5	0	0
With ICT Linear Algebra Courses are easier	13	22	0	0	0
There is no need to have ICT in Linear Algebra I – II	0	0	7	14	14
Designed instructional materials are helpful	23	6	6	0	0
I feel more confident with ICT	30	5	0	0	0
ICT enriches the lectures	29	6	0	0	0
I enjoy using ICT	33	2	0	0	0
I would like to have ICT also in other courses	19	11	4	1	0
Matrix algebra is the most important topic of Linear Algebra I and II	23	12	0	0	0
The concept of digonalization is one of the most important concept in Linear Algebra I- II	13	12	10	0	0

Table 2. Assessment of experimental dimension of ICT

Assessment of theoretical experimental dimension of ICT	**agree**	**Neutral**	**Disagree**
	(f)	(f)	(f)
Computations become easier with ICT	35	0	0
ICT is time-saving	35	0	0
ICT helps me to conserve my attention in matrix calculations	35	0	0
ICT provides a reason to understand the theory	29	6	0
ICT provides a mechanism for using the theorems	23	12	0
I feel like an unthinking button-pusher	0	3	32

Table 3. Assessment of algorithmic dimension of ICT

Assessment of algorithmic dimension of ICT	**agree**	**neutral**	**Disagree**
	(f)	(f)	(f)
Writing an algorithm helps me to deepen my perception of the concept	28	6	1
The flow charts for constructing an algorithm explains the processes	35	0	0
With this approach I improve my capabilities in making correct decisions of the results.	30	5	0
With an algorithm I can interpret the relations of reason-result	31	4	0
With an algorithm I repeat my course	29	6	0
A cause this approach I lost my attention to the courses	3	1	31
This approach is needless	1	1	33
Writing programs is difficult	12	13	10
Language of programming is easy to handle	9	13	13

6. CONCLUSION

The main findings in this case study indicate that the advanced calculators are valuable tools in teaching the concept of diagonalization which is one of the key and difficult topics in a Linear Algebra course at the undergraduate level. First, they offer assistance to the students for the computations by conserving their attention in the matrix calculations. Secondly, it is proved by the theoretical experimentation approach that these tools can be used in order to favor the development of mathematical knowledge and can act as a mediator. Finally, the process of writing an algorithm helps deepen the students' perception of the concept and improve their capabilities in making correct decisions. Once the students constructed the algorithms, the complete knowledge was achieved in their minds with all of the possible cases.

Another important contribution of these advanced personal tools was from a pedagogical aspect. With this case study, it is observed that the lectures have become more interactive and student-centered once the proper technology is integrated in teaching. In addition to their functional aspects, the use of advanced calculators has special pedagogical aspects. As it is noted in Heid, Sheets, and Matras (1990), with technology, new roles are assigned to the teachers as technical assistants, collaborators, facilators of student learning, and catalysts and this would pose a real challenge for the change of teaching experience.

This case study points out that the technology itself does not simplify the learning and it requires carefully designed teaching environments. In order to verify the robustness of this study, it is necessary to engage in a

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long-term study, including more observations on comparative results over the years, different technologies, and teaching practices with a larger sample of students. In that sense, this study can be seen as a first attempt to investigate how students learn an important subject within the Linear Algebra with an appropriate use of ICT.

The present study can be easily extended not only to the other subjects within Linear Algebra but also to the other undergraduate level mathematics courses such as calculus, differential equations, and probability and statistics. The integration of ICT tools to the entire curriculum can be promoted step by step by accumulating the relevant experiences in distinct fields. In that regard, setting up an open source web page involving several applications can be very useful. Another potential improvement can be sharing the relevant and successful ICT applications with other mathematics departments both locally in Turkey and possibly internationally. A discussion e-mail group can also be very useful in order to be able to develop new ideas about ICT integration to the entire curriculum of undergraduate mathematics education.

As technology advances and more students have access to new technologies, more opportunities would become available to help students learn and this will lead to new questions and treatments. As Laborde (2007) claims, the integration of ICT in teaching of mathematics involves long-term work including all kinds of relevant research, reflections and analyses, and it should be an integral part of teacher education and institutional assessment.

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